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FUNCTIONAL PROGRAMMING WITH APOMORPHISMS (CORECURSION)

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Abstract. In the mainstream categorical approach to typed (total) functional programming, functions with inductive source types defined by primitive recursion are called paramorphisms; the utility of primitive recursion as a scheme for defining functions in programming is well known. We draw attention to the dual notion of apomorphisms – functions with coinductive target types defined by primitive corecursion and show on examples that primitive corecursion is useful in programming.

Key words: typed (total) functional programming, categorical program calculation, (co)datatypes, (co)recursion forms.

1. INTRODUCTION

This paper is about the categorical approach to typed (total) functional programming where datatypes and codatatypes are modelled as initial algebras and terminal coalgebras. Our object of study is the notion of apomorphisms, which are functions with coinductive target types defined by primitive corecursion. Apomorphisms are the dual of paramorphisms, functions with inductive source types defined by primitive corecursion. The widely used term paramorphism was introduced by Meertens [1] ($\pi\alpha\rho\alpha$ – Greek preposition meaning "near to", "at the side of", "towards"), the term apomorphism is a novel invention of ours ($\alpha\pi o$ – Greek preposition meaning "apart from", "far from", "away from"). Our aim is to show that apomorphisms are a convenient tool for a declaratively thinking programmer, often more handy than anamorphisms, i.e. functions defined by

coiteration, in much the same way as paramorphisms are frequently superior to catamorphisms, i.e. functions defined by iteration.

The tradition of categorical functional programming that we follow in this paper started off with the Bird–Meertens formalism [2], which is a theory of the datatypes of lists. Malcolm [3] and Fokkinga [4] generalized the approach for arbitrary datatypes and codatatypes, inspired by the work of Hagino [5].

Geuvers [6] presents a thorough theoretical analysis of primitive recursion versus iteration and demonstrates that this readily dualizes into an analysis of primitive corecursion versus coiteration. In general, however, primitive corecursion has largely been overlooked in the theoretical literature, e.g. Fokkinga [4] ignores it, neither do we know of any papers containing interesting programming examples, the most probable reason being the young age of programming with codatatypes.

We shall proceed as follows. In Section 2, we give a short introduction to the categorical approach to (co)datatypes and (co)iteration and its application to program calculation. In Section 3, we introduce the scheme of primitive recursion for inductive types and prove several of its properties. Afterwards, by introducing primitive corecursion, we dualize these results for the case of coinductive types and show the usefulness of this scheme on several examples. Section 4 includes possible directions for further work and conclusions.

Proofs in this paper are carried out in the structured calculational proof style [7].

2. (CO)DATATYPES AND (CO)ITERATION

2.1. Preliminaries

Throughout the paper $\mathcal{C}$ is the default category, in which we shall assume the existence of finite products $(\times, 1)$ and coproducts $(+, 0)$, as well as the distributivity of products over coproducts (i.e. $\mathcal{C}$ is *distributive*). The typical example of a distributive category is **Sets** – the category of sets and total functions.

We make use of the following quite standard notation. Given two objects A, B, we write $\mathtt{fst} : A \times B \to A$ and $\mathtt{snd} : A \times B \to B$ to denote the left and right projections for the product $A \times B$. For $f : C \to A$ and $g : C \to B$, pairing is the unique morphism $\langle f, g \rangle : C \to A \times B$, such that $\mathtt{fst} \circ \langle f, g \rangle = f$ and $\mathtt{snd} \circ \langle f, g \rangle = g$. The left and right injections for the coproduct $A + B$ are $\mathtt{inl} : A \to A + B$ and $\mathtt{inr} : B \to A + B$. For $f : A \to C$ and $g : B \to C$, case analysis is the unique morphism $[f, g] : A + B \to C$, such that $[f, g] \circ \mathtt{inl} = f$ and $[f, g] \circ \mathtt{inr} = g$. Besides, given an object C, we have the unique morphism $!_C : C \to 1$. The inverse of the canonical map $[\mathtt{inl} \times \mathtt{id}_C, \mathtt{inr} \times \mathtt{id}_C] : (A \times C) + (B \times C) \to (A + B) \times C$ is denoted by $\mathtt{distr} : (A + B) \times C \to (A \times C) + (B \times C)$. Finally, given a predicate $p : A \to 1 + 1$, the guard $p? : A \to A + A$ is defined as $(\mathtt{snd} + \mathtt{snd}) \circ \mathtt{distr} \circ \langle p, \mathtt{id}_A \rangle$.

The identity functor is denoted by I. For a functor $F : \mathcal{C} \to \mathcal{C}$, its application to a morphism $f : A \to B$ is denoted by $Ff : FA \to FB$.

2.2. Dialgebras

Categorically, datatypes (of natural numbers, lists, etc.) are traditionally modelled by initial algebras and codatatypes (of conatural numbers, colists, streams, etc.) by terminal coalgebras. Hagino [5] introduced the notion of dialgebras which generalizes both concepts and is rich enough also to model many-sorted (co)algebras and bialgebras [4].

Definition 1. *Let $F, G : \mathcal{C} \to \mathcal{C}$ be endofunctors.*

- F, G-dialgebra *is a pair (A, φ), where A is an object and $\varphi : F\,A \to G\,A$ is a morphism of the default category $\mathcal{C}$.*
- *Let (A, φ) and (B, ψ) be F, G-dialgebras. A morphism $f : A \to B$ of $\mathcal{C}$ is an F, G-*homomorphism *from (A, φ) to (B, ψ) if the following diagram commutes:*

$$\begin{array}{ccc} F\,A & \xrightarrow{\varphi} & G\,A \\ \Big\downarrow{\scriptstyle F\,f} & & \Big\downarrow{\scriptstyle G\,f} \\ F\,B & \xrightarrow{\psi} & G\,B \end{array}$$

  (i.e. $Gf \circ \varphi = \psi \circ Ff$).

It is easy to verify that the composition of two F, G-homomorphisms is also an F, G-homomorphism. Moreover, the identity morphism $\mathtt{id}_A : A \to A$ is an F, G-homomorphism which is both a left and a right unit with respect to composition. It follows that dialgebras and homomorphisms form a category.

Definition 2. *Let $F, G : \mathcal{C} \to \mathcal{C}$ be endofunctors.*

- *A category of F, G-dialgebras $\mathcal{DiAlg}(F, G)$ is a category where objects are F, G-dialgebras and morphisms are F, G-homomorphisms. Composition and identities are inherited from $\mathcal{C}$. The categories of F-algebras and G-coalgebras are defined as $\mathcal{Alg}(F) = \mathcal{DiAlg}(F, I)$ and $\mathcal{CoAlg}(G) = \mathcal{DiAlg}(I, G)$, respectively.*
- *An F, G-dialgebra is* initial (*resp.* terminal) *if it is initial (resp. terminal) (as an object) in the category of F, G-dialgebras $\mathcal{DiAlg}(F, G)$. The initial F-algebra is denoted by $(\mu F, \mathtt{in}_F)$, the terminal G-coalgebra is denoted by $(\nu G, \mathtt{out}_G)$.*

Note that initial and terminal dialgebras may or may not exist. In fact, we even do not know any simple criteria for their existence in the general case of dialgebras. In case of algebras, an initial F-algebra is guaranteed to exist if a functor F is ω-(co)continuous. (A functor is ω-(co)continuous iff it preserves the (co)limits of ω-chains.) Dually, if a functor G is ω-continuous, then a terminal G-coalgebra

is guaranteed to exist. All polynomial functors (i.e. functors which are built up from products, sums, identity and constant functors) are ω-(co)continuous and, consequently, the corresponding initial algebras and terminal coalgebras exist. We do not pursue this question any further here and refer the interested reader to [4].

2.3. Initial algebras and catamorphisms

The existence of the initial F-algebra $(\mu F, \mathtt{in}_F)$ means that, for any F-algebra (C, ψ), there exists a unique homomorphism from $(\mu F, \mathtt{in}_F)$ to (C, ψ). Following Fokkinga [4], we denote this homomorphism by $(\!|\, \psi \,|\!)_F$, so $(\!|\, \psi \,|\!)_F : \mu F \to C$ is characterized by the universal property

$$f \circ \mathtt{in}_F = \psi \circ F\, f \quad \equiv \quad f = (\!|\, \psi \,|\!)_F \qquad \text{cata-CHARN.}$$

The type information is summarized in the following diagram:

$$\begin{array}{ccc} F\,\mu F & \xrightarrow{\mathtt{in}_F} & \mu F \\ {\scriptstyle F f}\downarrow & & \downarrow{\scriptstyle f} \\ F\,C & \xrightarrow{\psi} & C \end{array} .$$

Morphisms of the form $(\!|\, \psi \,|\!)_F$ are called *F-catamorphisms* (derived from the Greek preposition $\kappa\alpha\tau\alpha$ meaning "downwards"); the construction $(\!|\, \cdot \,|\!)_F$ is an iterator.

Example 1. Consider the datatype *Nat* of natural numbers, with constructor functions $zero : 1 \to Nat$ and $succ : Nat \to Nat$. It can be represented as an initial N-algebra $(Nat, [\, zero, succ \,]) = (\mu N, \mathtt{in}_N)$, where $N\,X = 1 + X$ and $N\,f = \mathtt{id}_1 + f$. Given any N-algebra $(C, [\, c, h \,])$, the catamorphism $f = (\!|\, [\, c, h \,] \,|\!)_N$ is the iteration $f(n) = h^n(c)$. For instance, the curried sum of two naturals can be defined as

$$add\ n = (\!|\, [\, n, succ \,] \,|\!)_N .$$

Example 2. The datatype $List_A$ of finite lists over a given set A, with constructor functions $nil : 1 \to List_A$ and $cons : A \times List_A \to List_A$, can be represented as an initial algebra $(List_A, [\, nil, cons \,]) = (\mu_{L_A}, \mathtt{in}_{L_A})$, where $L_A\,X = 1 + (A \times X)$ and $L_A\,f = \mathtt{id}_1 + (\mathtt{id}_A \times f)$. Given any L_A-algebra $(C, [\, c, h \,])$, the catamorphism $f = (\!|\, [\, c, h \,] \,|\!)_{L_A}$ is an application of the standard *fold* function on lists. For any function $g : A \to B$, the function $map\ g : List_A \to List_B$, for instance, can be defined as

$$map\ g = (\!|\, [\, nil, cons \circ (g \times \mathtt{id}_{List_A}) \,] \,|\!)_{L_A} .$$

Catamorphisms obey several nice laws, of which the *cancellation* law, the *reflection* law (also known as the identity law), and the *fusion* (or, promotion) law

are especially important for program calculation (the cancellation law describing a certain program execution step):

$$
\begin{array}{lll}
(\!|\psi|\!)_F \circ \mathtt{in}_F = \psi \circ F(\!|\psi|\!)_F & & \text{cata-CANCEL} \\
\mathtt{id}_{\mu F} = (\!|\mathtt{in}_F|\!)_F & & \text{cata-REFL} \\
f \circ \varphi = \psi \circ F f & \Rightarrow \quad f \circ (\!|\varphi|\!)_F = (\!|\psi|\!)_F & \text{cata-FUSION}\,.
\end{array}
$$

All three follow directly from the characterization cata-CHARN. These laws (originally for the special case of lists) form the heart of the Bird–Meertens formalism.

To show the usefulness of above-mentioned laws, we prove the following lemma (first recorded by Lambek [8]):

Lemma 1. *The morphism* $\mathtt{in}_F : F\,\mu F \to \mu F$ *is an isomorphism with the inverse* $\mathtt{in}_F^{-1} = (\!|F\,\mathtt{in}_F|\!)_F : \mu F \to F\,\mu F$.

Proof.

$$
\left[\begin{array}{ll}
 & \mathtt{in}_F \circ (\!|F\,\mathtt{in}_F|\!)_F \\
= & \quad \text{– cata-FUSION –} \\
 & (\!|\mathtt{in}_F|\!)_F \\
= & \quad \text{– cata-REFL –} \\
 & \mathtt{id}_{\mu F}
\end{array}\right.
\qquad
\left[\begin{array}{ll}
 & (\!|F\,\mathtt{in}_F|\!)_F \circ \mathtt{in}_F \\
= & \quad \text{– cata-CANCEL –} \\
 & F\,\mathtt{in}_F \circ F\,(\!|F\,\mathtt{in}_F|\!)_F) \\
= & \quad \text{– } F \text{ functor –} \\
 & F\,(\mathtt{in}_F \circ (\!|F\,\mathtt{in}_F|\!)_F) \\
= & \quad \text{– above –} \\
 & F\,\mathtt{id}_{\mu F} \\
= & \quad \text{– } F \text{ functor –} \\
 & \mathtt{id}_{F\,\mu F}
\end{array}\right.
$$

2.4. Terminal coalgebras and anamorphisms

The existence of the terminal G-coalgebra $(\nu G, \mathtt{out}_G)$ means that for any other G-coalgebra (C, φ) there exists a unique homomorphism from (C, φ) to $(\nu G, \mathtt{out}_G)$. This homomorphism is usually denoted by $[\!|\varphi|\!]_G$, so $[\!|\varphi|\!]_G : C \to \nu G$ is characterized by the universal property

$$
\mathtt{out}_G \circ f = G f \circ \varphi \quad \equiv \quad f = [\!|\varphi|\!]_G \qquad \text{ana-CHARN}\,.
$$

The type information is summarized in the following diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{\varphi} & G\,C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G f} \\
\nu G & \xrightarrow{\mathtt{out}_G} & G\,\nu G\,.
\end{array}
$$

Morphisms of the form $[\!|\varphi|\!]_G$ are called *G-anamorphisms* (derived from the Greek preposition $\alpha\nu\alpha$ meaning "upwards"; the name is due to Meijer [9]), and the

construction $[\![\cdot]\!]_G$ is a coiterator. Like catamorphisms, anamorphisms have various properties including the following cancellation, reflection, and fusion laws:

$$\begin{array}{lll} \mathtt{out}_G \circ [\![\varphi]\!]_G = G\,[\![\varphi]\!]_G \circ \varphi & & \text{ana-CANCEL} \\ \mathtt{id}_{\nu G} = [\![\mathtt{out}_G]\!]_G & & \text{ana-REFL} \\ \psi \circ f = G\,f \circ \varphi & \Rightarrow \quad [\![\psi]\!]_G \circ f = [\![\varphi]\!]_G & \text{ana-FUSION.} \end{array}$$

Also, $\mathtt{out}_G$ is an isomorphism with the inverse $\mathtt{out}_G^{-1} = [\![G\,\mathtt{out}_G]\!]_G$.

Example 3. Consider the codatatype $Stream_A$ of streams (infinite lists) over A, with destructor functions $head : Stream_A \to A$ and $tail : Stream_A \to Stream_A$. It can be represented as a terminal coalgebra $(Stream_A, \langle head, tail \rangle) = (\nu S_A, \mathtt{out}_{S_A})$, where $S_A\,X = A \times X$ and $S_A\,f = \mathtt{id}_A \times f$. Anamorphisms are used to construct concrete streams. For instance, the stream of natural numbers $nats : 1 \to Stream_{Nat}$ and "zipping" of two streams $zip : Stream_A \times Stream_B \to Stream_{A \times B}$ can be defined as follows:

$$\begin{array}{l} nats = [\![\langle \mathtt{id}_{Nat}, succ \rangle]\!]_{S_{Nat}} \circ zero \\ zip = [\![\langle \mathtt{fst} \times \mathtt{fst}, \mathtt{snd} \times \mathtt{snd} \rangle \circ (\mathtt{out}_{S_A} \times \mathtt{out}_{S_B})]\!]_{S_{A \times B}}. \end{array}$$

Example 4. The codatatype of colists (possibly infinite lists) $List'_A$ can be represented as a terminal coalgebra $(\nu L_A, \mathtt{out}_{L_A})$ of the functor $L_A(X) = 1 + A \times X$. Given a predicate $empty : List'_A \to 1 + 1$ and two partial functions $head' : List'_A \to A$ and $tail' : List'_A \to List'_A$, the terminal coalgebra morphism $\mathtt{out}_{L_A}$ can be defined as $\mathtt{out}_{L_A} = (! + \langle head', tail' \rangle) \circ empty?$. Anamorphisms for $List'_A$ correspond to applications of the *unfold* function from functional programming.

3. PRIMITIVE (CO)RECURSION

Clearly, not every morphism $f : \mu F \to C$ is a catamorphism by itself, which means that the results of the previous section cannot be applied as they stand. For instance, the factorial function specified by equations

$$\begin{array}{lcl} fact(0) & = & 1 \\ fact(n+1) & = & (n+1) * fact(n) \end{array}$$

is not a catamorphism. The problem here is that, to compute the value of the factorial for a given argument, one needs not only its value for the previous argument, but also the previous argument itself. Meertens [1] has shown that any function which can be characterized similarly to the factorial function is definable as the composition of a left projection and a catamorphism. The relevant result is the following.

Lemma 2. *Given* $f : \mu F \to C$ *and* $\psi : F\,(C \times \mu F) \to C$,

$$f \circ \mathtt{in}_F = \psi \circ F\,\langle f, \mathtt{id}_{\mu F} \rangle \equiv f = \mathtt{fst} \circ (\!|\, \langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd} \rangle \,|\!)_F.$$

Proof.

$$
\begin{array}{l}
\lceil\ \rhd\ \ f \circ \mathtt{in}_F = \psi \circ F\,\langle f, \mathtt{id}_{\mu F}\rangle \\
\hline
\quad f \\
= \quad \text{– pairing –} \\
\quad \mathtt{fst} \circ \langle f, \mathtt{id}_{\mu F}\rangle \\
= \quad \text{– cata-CHARN –} \\
\qquad \lceil\ \ \langle f, \mathtt{id}_{\mu F}\rangle \circ \mathtt{in}_F \\
\qquad = \quad \text{– pairing –} \\
\qquad\quad \langle f \circ \mathtt{in}_F, \mathtt{in}_F\rangle \\
\qquad = \quad \text{– } F \text{ functor –} \\
\qquad\quad \langle f \circ \mathtt{in}_F, \mathtt{in}_F \circ F\,\mathtt{id}_{\mu F}\rangle \\
\qquad = \quad \text{– pairing –} \\
\qquad\quad \langle f \circ \mathtt{in}_F, \mathtt{in}_F \circ F\,(\mathtt{snd} \circ \langle f, \mathtt{id}_{\mu F}\rangle)\rangle \\
\qquad = \quad \text{– } \lhd, F \text{ functor –} \\
\qquad\quad \langle \psi \circ F\,\langle f, \mathtt{id}_{\mu F}\rangle, \mathtt{in}_F \circ F\,\mathtt{snd} \circ F\,\langle f, \mathtt{id}_{\mu F}\rangle\rangle \\
\qquad = \quad \text{– pairing –} \\
\qquad\quad \langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle \circ F\,\langle f, \mathtt{id}_{\mu F}\rangle \\
\quad \mathtt{fst} \circ (\!|\,\langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle\,|\!)_F
\end{array}
$$

$$
\begin{array}{l}
\lceil\ \rhd\ \ f = \mathtt{fst} \circ (\!|\,\langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle\,|\!)_F \\
\hline
\quad f \circ \mathtt{in}_F \\
= \quad \text{– } \lhd \text{ –} \\
\quad \mathtt{fst} \circ (\!|\,\langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle\,|\!)_F \circ \mathtt{in} \\
= \quad \text{– cata-CANCEL –} \\
\quad \mathtt{fst} \circ \langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle \circ F\,(\!|\,\langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle\,|\!)_F \\
= \quad \text{– pairing, 2x –} \\
\quad \psi \circ F\,\langle \mathtt{fst} \circ (\!|\,\langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle\,|\!)_F, \mathtt{snd} \circ (\!|\,\langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle\,|\!)_F\rangle \\
= \quad \text{– } \lhd, \text{cata-FUSION –} \\
\qquad \lceil\ \ \mathtt{snd} \circ \langle \psi, \mathtt{in}_F \circ F\,\mathtt{snd}\rangle \\
\qquad = \quad \text{– pairing –} \\
\qquad\quad \mathtt{in}_F \circ F\,\mathtt{snd} \\
\quad \psi \circ F\,\langle f, (\!|\,\mathtt{in}_F\,|\!)_F\rangle \\
= \quad \text{– cata-REFL –} \\
\quad \psi \circ F\,\langle f, \mathtt{id}_{\mu F}\rangle
\end{array}
$$

The left-hand side of the equivalence corresponds exactly to the specification by primitive recursion. For instance, for the factorial function specification above, it reads

$$
\begin{array}{l}
\quad fact \circ \mathtt{in}_N \\
= \quad [\,succ \circ zero, mult \circ (\mathtt{id}_{Nat} \times succ)\,] \circ N\,\langle fact, \mathtt{id}_{Nat}\rangle \\
= \quad [\,succ \circ zero, mult \circ (fact \times succ)\,]
\end{array}
$$

and consequently we get the definition of the factorial

$$fact = \mathtt{fst} \circ (\!|\, \langle\, [\, succ \circ zero, mult \circ (\mathtt{id}_{Nat} \times succ)\,], \mathtt{in}_N \circ N\, \mathtt{snd}\,\rangle \,|\!)_N .$$

From the lemma above it follows that at least every primitive recursive function can be represented using a catamorphism as the only recursive construction. In the presence of exponentials, one can even define Ackermann's function as a (higher-order) catamorphism, so the expressive power of the "language of catamorphisms" is bigger than the class of primitively recursive functions. In fact, Howard [10] has shown that the functions expressible in simply typed λ-calculus extended with inductive and coinductive types are precisely those provably total in the logic $ID_{<\omega}$ (the first-order arithmetic augmented by finitely-iterated inductive definitions).

3.1. Paramorphisms

To make programming and program reasoning easier, let us introduce a new construction and find out its properties. For any morphism $\psi : F\,(C \times \mu F) \to C$, define a morphism $(\!|\, \psi \,|\!)_F : \mu F \to C$ by letting

$$(\!|\, \psi \,|\!)_F = \mathtt{fst} \circ (\!|\, \langle\, \psi, \mathtt{in}_F \circ F\, \mathtt{snd}\,\rangle \,|\!)_F \qquad \text{para-DEF}.$$

Morphisms of the form $(\!|\, \psi \,|\!)_F$ are called *F-paramorphisms* (the name is due to Meertens [1]). The construction $(\!|\, \cdot \,|\!)_F$ is a primitive recursor.

From Lemma 2 we get the characterization of paramorphisms by the following universal property:

$$f \circ \mathtt{in}_F = \psi \circ F\,\langle f, \mathtt{id}_{\mu F}\,\rangle \;\equiv\; f = (\!|\, \psi \,|\!)_F \qquad \text{para-CHARN}.$$

The type information is summarized in the following diagram:

$$\begin{array}{ccc} F\,\mu F & \xrightarrow{\ \mathtt{in}\ } & \mu F \\ {\scriptstyle F\langle f, \mathtt{id}_{\mu F}\rangle}\downarrow & & \downarrow{\scriptstyle f} \\ F\,(C \times \mu F) & \xrightarrow{\ \psi\ } & C \end{array} .$$

Example 5. The factorial function can be defined as a paramorphism:

$$fact = (\!|\, [\, succ \circ zero, mult \circ (\mathtt{id}_{Nat} \times succ)\,] \,|\!)_N .$$

The calculational properties of paramorphisms are similar to those of catamorphisms. In particular, we have "paramorphic" versions of the cancellation, reflection, and fusion laws:

$$(\!|\, \psi \,|\!)_F \circ \mathtt{in}_F = \psi \circ F\,\langle\, (\!|\, \psi \,|\!)_F, \mathtt{id}_{\mu F}\,\rangle \qquad \text{para-CANCEL}$$

$$\mathtt{id}_{\mu F} = (\!|\, \mathtt{in}_F \circ F\, \mathtt{fst} \,|\!)_F \qquad \text{para-REFL}$$

$$f \circ \varphi = \psi \circ F\,(f \times \mathtt{id}_{\mu F}) \;\Rightarrow\; f \circ (\!|\, \varphi \,|\!)_F = (\!|\, \psi \,|\!)_F \qquad \text{para-FUSION}.$$

The reflection law is proved by the following calculation:

$$
\begin{array}{ll}
 & \mathtt{id}_{\mu F} \\
= & \quad \text{– para-CHARN –} \\
 & \quad \begin{array}{ll}
 & \mathtt{in}_F \\
= & \quad \text{– } F \text{ functor –} \\
 & \mathtt{in}_F \circ F\,\mathtt{id}_{\mu F} \\
= & \quad \text{– pairing –} \\
 & \mathtt{in}_F \circ F\,(\mathtt{fst} \circ \langle\, \mathtt{id}_{\mu F}, \mathtt{id}_{\mu F} \,\rangle) \\
= & \quad \text{– } F \text{ functor –} \\
 & \mathtt{in}_F \circ F\,\mathtt{fst} \circ F\,\langle\, \mathtt{id}_{\mu F}, \mathtt{id}_{\mu F} \,\rangle
\end{array} \\
 & \quad (\!|\, \mathtt{in}_F \circ F\,\mathtt{fst} \,|\!)_F \qquad .
\end{array}
$$

The fusion law is proved as follows:

$$
\begin{array}{ll}
 & \triangleright \quad f \circ \varphi = \psi \circ F\,(f \times \mathtt{id}_{\mu F}) \\
 & \hline \\
 & f \circ (\!|\, \varphi \,|\!)_F \\
= & \quad \text{– para-CHARN –} \\
 & \quad \begin{array}{ll}
 & f \circ (\!|\, \varphi \,|\!)_F \circ \mathtt{in}_F \\
= & \quad \text{– para-CANCEL –} \\
 & f \circ \varphi \circ F\,\langle\, (\!|\, \varphi \,|\!)_F, \mathtt{id}_{\mu F} \,\rangle \\
= & \quad \text{– } \triangleleft \text{ –} \\
 & \psi \circ F\,(f \times \mathtt{id}_{\mu F}) \circ F\,\langle\, (\!|\, \varphi \,|\!)_F, \mathtt{id}_{\mu F} \,\rangle \\
= & \quad \text{– } F \text{ functor –} \\
 & \psi \circ F\,((f \times \mathtt{id}_{\mu F}) \circ \langle\, (\!|\, \varphi \,|\!)_F, \mathtt{id}_{\mu F} \,\rangle) \\
= & \quad \text{– pairing –} \\
 & \psi \circ F\,\langle\, f \circ (\!|\, \varphi \,|\!)_F, \mathtt{id}_{\mu F} \,\rangle
\end{array} \\
 & \quad (\!|\, \psi \,|\!)_F \qquad .
\end{array}
$$

By definition, every paramorphism is the composition of left projection and a catamorphism. In converse, paramorphisms can be viewed as a generalization of catamorphisms, in the sense that every catamorphism is definable as a certain paramorphism:

$$(\!|\, \psi \,|\!)_F = (\!|\, \psi \circ F\,\mathtt{fst} \,|\!)_F \qquad \text{para-CATA.}$$

This property can be verified by the following calculation:

$$
\begin{array}{ll}
 & (\!|\psi|\!)_F \\
= & \quad \text{– para-CHARN –} \\
 & \quad \left[\begin{array}{ll}
 & (\!|\psi|\!)_F \circ \mathtt{in}_F \\
= & \quad \text{– cata-CANCEL –} \\
 & \psi \circ F\,(\!|\psi|\!)_F \\
= & \quad \text{– pairing –} \\
 & \psi \circ F\,(\mathtt{fst} \circ \langle\, (\!|\psi|\!)_F, \mathtt{id}_{\mu F} \,\rangle) \\
= & \quad \text{– } F \text{ functor –} \\
 & \psi \circ F\,\mathtt{fst} \circ F\,\langle\, (\!|\psi|\!)_F, \mathtt{id}_{\mu F} \,\rangle
\end{array}\right] \\
 & (\!|\psi \circ F\,\mathtt{fst}|\!)_F \quad .
\end{array}
$$

3.2. Apomorphisms

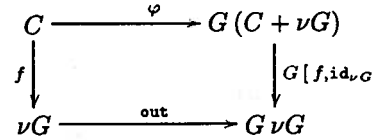
Let us now dualize everything we know about paramorphisms. For any morphism $\varphi : C \to G\,(C + \nu G)$, define a morphism $[\!(\varphi)\!] : C \to \nu G$ as the composition of a certain anamorphism and left injection:

$$[\!(\varphi)\!]_G = [\![\, [\varphi, G\,\mathtt{inr} \circ \mathtt{out}_G \,]\!]_G \circ \mathtt{inl} \qquad \text{apo-DEF.}$$

Let us agree to call morphisms of the form $[\!(\varphi)\!]_G$ *G-apomorphisms*. The construction $[\!(\cdot)\!]_G$ is, of course, a primitive corecursor. The characterizing universal property for apomorphisms is the following:

$$\mathtt{out}_G \circ f = G\,[f, \mathtt{id}_{\nu G}] \circ \varphi \quad \equiv \quad f = [\!(\varphi)\!]_G \qquad \text{apo-CHARN.}$$

The type information is summarized in the following diagram:



The laws for apomorphisms are just the duals of those for paramorphisms:

$$
\begin{array}{lll}
\mathtt{out}_G \circ [\!(\varphi)\!]_G = G\,[\![\, [\!(\varphi)\!]_G, \mathtt{id}_{\nu G} \,]\!] \circ \varphi & & \text{apo-CANCEL} \\
\mathtt{id}_{\nu G} = [\!(G\,\mathtt{inl} \circ \mathtt{out}_G)\!]_G & & \text{apo-REFL} \\
\psi \circ f = G\,(f + \mathtt{id}_{\nu G}) \circ \varphi & \Rightarrow \quad [\!(\psi)\!]_G \circ f = [\!(\varphi)\!]_G & \text{apo-FUSION.}
\end{array}
$$

As paramorphisms are a generalization of catamorphisms, apomorphisms are a generalization of anamorphisms:

$$[\!(\varphi)\!]_G = [\!(G\,\mathtt{inl} \circ \varphi)\!]_G \qquad \text{apo-ANA.}$$

Example 6. Consider the function $maphd\ f : Stream_A \to Stream_A$, which maps a function $f : A \to A$ onto the head of a stream and leaves the tail of the stream unchanged. Using anamorphisms, it can be defined as follows:

$$maphd\ f = [\![\, [\, \langle f \circ head, \mathtt{inr} \circ tail \rangle, \langle head, \mathtt{inr} \circ tail \rangle \,] \,]\!]_{S_A} \circ \mathtt{inl}.$$

Clearly, such definition is not very convenient, as the tail of the resulting stream, which is unchanged, is also constructed element by element. At the same time, using apomorphisms, the definition looks as follows:

$$\begin{array}{rcl} maphd\ f & = & [\!(\, (f \times \mathtt{inr}) \circ \mathtt{out}_{S_A} \,)\!]_{S_A} \\ & = & [\!(\, \langle f \circ head, \mathtt{inr} \circ tail \rangle \,)\!]_{S_A}\,. \end{array}$$

Now, the tail of the stream is returned directly without any inspection inside of it.

Example 7. Assume that the set A is ordered linearly by a relation $(\leq) : A \times A \to 1+1$. The function $insert : A \times Stream_A \to Stream_A$ that inserts a given element into a sorted stream so that the result will also be sorted is specified as follows:

$$\langle head, tail \rangle \circ insert \circ \langle x, ys \rangle$$

$$= \begin{cases} \langle x, ys \rangle & \text{if } x \leq head \circ ys \\ \langle head \circ ys, insert \circ \langle x, tail \circ ys \rangle \rangle & \text{otherwise} \end{cases}.$$

In this specification, the input stream is traversed and copied to the output stream elementwise to the point at which the input element is inserted; the remainder of the input stream is entered to the output stream as one whole. Formally, this idea is captured by the following categorical definition:

$$insert \circ \langle x, ys \rangle = [\!(\, [\, \langle x \circ !, \mathtt{inr} \rangle, (\mathtt{id}_A \times \mathtt{inl}) \,] \circ test(x) \,)\!]_{S_A} \circ ys\,,$$

where $test(x) : Stream_A \to Stream_A + S_A\ Stream_A$ is

$$test(x) = (\mathtt{fst} + \mathtt{snd}) \circ ((\leq) \circ \langle x \circ !, \mathtt{fst} \circ \mathtt{snd} \rangle)? \circ \langle \mathtt{id}_{Stream_A}, \mathtt{out}_{S_A} \rangle.$$

Example 8. Consider the function on colists $concat : List'_A \times List'_A \to List'_A$ which concatenates two colists. It is naturally representable as an apomorphism which copies the first colist and, while arriving at the end of it, returns the second one:

$$concat \circ \langle x, y \rangle = [\!(\, [\, L_A \mathtt{inr} \circ \mathtt{out}_{L_A} \circ y, \mathtt{inr} \,] \circ L_A \mathtt{inl} \circ \mathtt{out}_{L_A} \,)\!]_{L_A} \circ x.$$

In order to illustrate the utility of the laws introduced above, we show that the concatenation of colists is associative:

$$concat \circ \langle\, concat \circ \langle x, y \rangle, z \,\rangle = concat \circ \langle x, concat \circ \langle y, z \rangle \,\rangle.$$

First, let us introduce a shorthand notation for the apomorphism occurring in the definition of the function *concat*:

$$c(y) = (\!|\, [\, L_A \mathtt{inr} \circ \mathtt{out}_{L_A} \circ y, \mathtt{inr}\,] \circ L_A \mathtt{inl} \circ \mathtt{out}_{L_A} \,|\!)_{L_A}.$$

So, the definition of *concat* is equivalent to $concat \circ \langle x, y \rangle = c(y) \circ x$, and the equation we want to prove looks as follows:

$$c(z) \circ c(y) \circ x = c(c(z) \circ y) \circ x\,.$$

The proof is carried out according to the following informal plan. First, we twice use the apomorphism characterization to "open" the bodies of both apomorphisms on the left-hand side of the equation. Then we "push" all subexpressions as far to the right as possible and simplify the resulting expression. Finally, we "pull" the simplified subexpressions back to the left and use again the characterization of apomorphisms to "close" their bodies. When "pushing" subexpressions through the case operation, we shall repeatedly use the following intermediate result about the functor L_A:

$$L_A\,[\,f, \mathtt{id}_{List'_A}\,] \circ [\,L_A\,\mathtt{inr} \circ g, \mathtt{inr}\,] \circ L_A\,\mathtt{inl} = [\,g, \mathtt{inr}\,] \circ L_A\,f. \qquad (*)$$

The validity of the equation $(*)$ follows directly from the definition of the functor L_A and the properties of the coproduct:

$$\left[\begin{array}{ll}
 & L_A\,[\,f, \mathtt{id}_{List'_A}\,] \circ [\,L_A\,\mathtt{inr} \circ g, \mathtt{inr}\,] \circ L_A\,\mathtt{inl} \\
= & \quad \text{– case analysis –} \\
 & [\,L_A\,[\,f, \mathtt{id}_{List'_A}\,] \circ L_A \mathtt{inr} \circ g, L_A\,[\,f, \mathtt{id}_{List'_A}\,] \circ \mathtt{inr}\,] \circ L_A\,\mathtt{inl} \\
= & \quad \text{– } L_A \text{ functor, case analysis, } L_A \text{ functor –} \\
 & [\,g, L_A\,[\,f, \mathtt{id}_{List'_A}\,] \circ \mathtt{inr}\,] \circ L_A\,\mathtt{inl} \\
= & \quad \text{– def. of } L_A \text{, case analysis –} \\
 & [\,g, \mathtt{inr} \circ (\mathtt{id}_A \times [\,f, \mathtt{id}_{List'_A}\,])\,] \circ L_A\,\mathtt{inl} \\
= & \quad \text{– case analysis –} \\
 & [\,g, \mathtt{inr}\,] \circ (\mathtt{id}_1 + (\mathtt{id}_A \times [\,f, \mathtt{id}_{List'_A}\,])) \circ L_A\,\mathtt{inl} \\
= & \quad \text{– def. of } L_A \text{ –} \\
 & [\,g, \mathtt{inr}\,] \circ L_A\,[\,f, \mathtt{id}_{List'_A}\,] \circ L_A\,\mathtt{inl} \\
= & \quad \text{– } L_A \text{ functor, case analysis –} \\
 & [\,g, \mathtt{inr}\,] \circ L_A\,f
\end{array}\right.$$

Now, the proof of associativity is the following:

$$
\begin{array}{l}
\Big[\quad c(z) \circ c(y) \circ x \\
= \quad \text{– def. of } c\text{, apo-CHARN –} \\
\quad \Big[\quad \mathtt{out}_{L_A} \circ c(z) \circ c(y) \\
\quad = \quad \text{– def. of } c\text{, apo-CANCEL –} \\
\quad\quad L_A\,[\,c(z), \mathtt{id}_{List'_A}\,] \circ [\,L_A\,\mathtt{inr} \circ \mathtt{out}_{L_A} \circ z, \mathtt{inr}\,] \circ L_A\,\mathtt{inl} \\
\quad\quad\quad\quad \circ \mathtt{out}_{L_A} \circ c(y) \\
\quad = \quad \text{– (*) –} \\
\quad\quad [\,\mathtt{out}_{L_A} \circ z, \mathtt{inr}\,] \circ L_A\,(c(z)) \circ \mathtt{out}_{L_A} \circ c(y) \\
\quad = \quad \text{– def. of } c\text{, apo-CANCEL –} \\
\quad\quad [\,\mathtt{out}_{L_A} \circ z, \mathtt{inr}\,] \circ L_A\,(c(z)) \circ L_A[\,c(y), \mathtt{id}_{List'_A}\,] \circ \\
\quad\quad\quad\quad \circ [\,L_A\,\mathtt{inr} \circ \mathtt{out}_{L_A} \circ y, \mathtt{inr}\,] \circ L_A\,\mathtt{inl} \circ \mathtt{out}_{L_A} \\
\quad = \quad \text{– (*) –} \\
\quad\quad [\,\mathtt{out}_{L_A} \circ z, \mathtt{inr}\,] \circ L_A\,(c(z)) \circ [\,\mathtt{out}_{L_A} \circ y, \mathtt{inr}\,] \circ L_A\,(c(y)) \\
\quad\quad\quad\quad \circ \mathtt{out}_{L_A} \\
\quad = \quad \text{– case analysis, def. of } L_A \text{ –} \\
\quad\quad [\,\mathtt{out}_{L_A} \circ z, \mathtt{inr}\,] \circ [\,L_A\,(c(z)) \circ \mathtt{out}_{L_A} \circ y, \mathtt{inr}\,] \circ L_A\,(c(z)) \\
\quad\quad\quad\quad \circ L_A\,(c(y)) \circ \mathtt{out}_{L_A} \\
\quad = \quad \text{– case analysis, } L_A \text{ functor –} \\
\quad\quad [[\,\mathtt{out}_{L_A} \circ z, \mathtt{inr}\,] \circ L_A\,(c(z)) \circ \mathtt{out}_{L_A} \circ y, \mathtt{inr}\,] \circ L_A\,(c(z) \\
\quad\quad\quad\quad \circ c(y)) \circ \mathtt{out}_{L_A} \\
\quad = \quad \text{– (*) –} \\
\quad\quad [L_A\,[\,c(z), \mathtt{id}_{List'_A}\,] \circ [\,L_A\,\mathtt{inr} \circ \mathtt{out}_{L_A} \circ z, \mathtt{inr}\,] \circ L_A\,\mathtt{inl} \\
\quad\quad\quad\quad \circ \mathtt{out}_{L_A} \circ y,\ \mathtt{inr}] \circ L_A\,(c(z) \circ c(y)) \circ \mathtt{out}_{L_A} \\
\quad = \quad \text{– apo-CANCEL, def. of } c \text{ –} \\
\quad\quad [\,\mathtt{out}_{L_A} \circ c(z) \circ y, \mathtt{inr}\,] \circ L_A\,(c(z) \circ c(y)) \circ \mathtt{out}_{L_A} \\
\quad = \quad \text{– (*) –} \\
\quad\quad L_A\,[\,c(z) \circ c(y), \mathtt{id}_{List'_A}\,] \circ [\,L_A\,\mathtt{inr} \circ \mathtt{out}_{L_A} \circ c(z) \circ y, \mathtt{inr}\,] \\
\quad\quad\quad\quad \circ L_A \mathtt{inl} \circ \mathtt{out}_{L_A} \\
\quad c(c(z) \circ y) \circ x
\end{array}
$$

4. CONCLUSIONS AND FURTHER WORK

We have described the notion of primitive corecursion for coinductive types that is dual to primitive recursion for inductive types. Both of them are generalizations of more basic operations – iteration and coiteration, respectively. While the value of an iterative function for a given argument depends solely on the values for its immediate subparts, the value of a primitive recursive function may additionally depend on these immediate subparts directly. Dually, the argument of a coiterative function for a value may only determine the arguments for the immediate subparts of the value, whereas the argument of a primitive corecursive function may alternatively determine these immediate subparts directly. Primitive corecursion,

being more permissive than coiteration, makes it easier to write programs and to prove properties about programs.

Codatatypes and related notions such as coinduction and bisimulation are widely used in the analysis of processes specifiable by transition systems or state machines [11]. If processes are understood as functions from states to behaviours, sequential composition of two processes becomes a natural example of a function elegantly definable as an apomorphism. This leads us to believe that apomorphisms may turn out a viable construction in the modelling of processes. Checking out this conjecture is one possible direction for continuing the work reported here.

In [12], we studied programming with (co)inductive types in the setting of intuitionistic natural deduction. Besides simple (co)iteration and primitive (co)recursion, we also considered course-of-value (co)iteration. A categorical treatment of these schemes is a subject for further work.

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FUNKTSIONAALPROGRAMMEERIMINE APOMORFISMIDEGA (KOREKURSIOONIGA)

Varmo VENE ja Tarmo UUSTALU

On käsitletud kategooriate teoreetilist lähenemist tüüpidega funktsionaalprogrammeerimises, kus andmetüüpe ja koandmetüüpe modelleeritakse vastavalt initsiaalsete algebratena ning terminaalsete koalgebratena. Peamiseks uurimisobjektiks on apomorfismid kui funktsioonid, mis on defineeritud lihtkorekursiooniskeemi abil. Apomorfismid on duaalsed lihtrekursiooniskeemi abil defineeritavate funktsioonidega – paramorfismidega. Sel ajal kui paramorfismid on leidnud laialdast käsitlemist nii teoorias kui ka praktikas, on apomorfismid seni suhteliselt tagasihoidlikku tähelepanu pälvinud. Siinses töös on näidatud, et apomorfismidel on mitmed olulised algebralised omadused, mis lihtsustavad korekursiivselt defineeritavate funktsioonide formaalset käsitlemist. Samuti on vaadeldud reaalseid näiteid koandmetüüpidega funktsionaalprogrammeerimisest, kus apomorfismid (korekursioon) on kasulikud konstruktsioonid.